量子光学笔记

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写在前面：本篇笔记主要参考 Scully，补充了相关内容的程序模拟。不完全 follow 课件，用于复习时请自行跳过某些内容。

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1 电磁场的量子理论

1.1 电磁场的量子化

1.1.1 方法一：正则量子化方法

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(1) 构造系统的 Lagrangian，计算系统的 Hamiltonian；

(2) 规定算符及其对易关系；

(3) 寻求适当的正交归一本征函数完全集；

(4) 对场算符作展开，从而引进“粒子”的产生和湮灭算符；

(5) 转到粒子数表象，写出结果。

电磁场的 Lagrangian 密度

$$\mathcal{L}=\frac{1}{2}\left(\varepsilon_0\vec{E}^2-\frac{1}{\mu_0}\vec{B}^2\right)=\frac{1}{2}\left[\varepsilon_0\left(\frac{\partial\vec{A}}{\partial t}\right)^2-\frac{1}{\mu_0}\left(\nabla\times\vec{A}\right)^2\right] \tag{1}$$

电磁场系统的广义坐标为 $\vec{A}(\vec{r},t)$，正则动量为 $\vec{P}$

$$\vec{P}=\frac{\partial\mathcal{L}}{\partial\dot{\vec{A}}}=\varepsilon_0\frac{\partial\vec{A}}{\partial t}=-\varepsilon_0\vec{E} \tag{2}$$

于是，Hamiltonian 密度

$$\mathcal{H}=\vec{P}\cdot\dot{\vec{A}}-\mathcal{L}=\frac{\vec{P}^2}{2\varepsilon_0}+\frac{1}{2\mu_0}\left(\nabla\times\vec{A}\right)^2 \tag{3}$$

将广义坐标 $\vec{A}$ 转换为场算符 $\hat{\vec{A}}$，将正则动量 $\vec{P}$ 转换为正则动量算符 $\hat{\vec{P}}$，规定其对易关系

$$\left[\hat{\vec{A}}_i(\vec{r},t),\hat{\vec{A}}_j(\vec{r}',t)\right]=0 \qquad \left[\hat{\vec{P}}_i(\vec{r},t),\hat{\vec{P}}_j(\vec{r}',t)\right]=0 \qquad \left[\hat{\vec{A}}_i(\vec{r},t),\hat{\vec{P}}_j(\vec{r}',t)\right]=i\hbar\delta_{ij}\delta(\vec{r}-\vec{r}')$$

接下来需要寻求适当的正交归一本征函数基矢，将场算符展开。对于一个大而均匀的系统，我们很自然地采用满足周期性边界条件的箱归一平面波完全集作为展开基矢；对于原子中相互作用的电子系统，通常采用单粒子库仑波函数完全集作为展开基矢；对于晶格点阵中运动的粒子，通常选取周期性势场中 Bloch 波函数完全集作为展开基矢。注意，**无论选择何种基矢，一旦选定，粒子数表象中的“粒子”即被赋予相应的物理含义**。这里我们选取以下本征函数完全集

$$\varphi(\vec{r})=\frac{1}{\sqrt{V}}e^{i\vec{k}\cdot\vec{r}} \tag{4}$$

于是

$$\hat{\vec{A}}(\vec{r},t)=\frac{1}{\sqrt{V}}\sum_{\vec{k}}\vec{e}_{\vec{k},\sigma}\left[\hat{A}_{\vec{k},\sigma}(t)e^{i\vec{k}\cdot\vec{r}}+\hat{A}^{\dagger}_{\vec{k},\sigma}(t)e^{-i\vec{k}\cdot\vec{r}}\right] \tag{5}$$

电磁场为实数矢量场，故矢势也是实数矢量场，即 $\vec{A}=\vec{A}^{\dagger}$，将广义坐标算符 $\hat{\vec{A}}$ 写成如下形式

$$\hat{\vec{A}}(\vec{r},t)=\sum_{\vec{k},\sigma}\vec{e}_{\vec{k},\sigma}\beta_{\vec{k}}\hat{C}_{\vec{k},\sigma}e^{-i\omega_{\vec{k}}t+i\vec{k}\cdot\vec{r}}+\text{H.C.} \tag{6}$$

其中 $\beta_{\vec{k}} = \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_{\vec{k}}}}$ 具有矢势的量纲。$\hat{C}_{\vec{k},\sigma}$ 和 $\hat{C}^\dagger_{\vec{k},\sigma}$ 分别为光子的湮灭和产生算符，满足波色子对易关系

$$\left[\hat{C}_{\vec{k},\sigma}, \hat{C}_{\vec{k}',\sigma'}\right] = 0 \qquad \left[\hat{C}^\dagger_{\vec{k},\sigma}, \hat{C}^\dagger_{\vec{k}',\sigma'}\right] = 0 \qquad \left[\hat{C}_{\vec{k},\sigma}, \hat{C}^\dagger_{\vec{k}',\sigma'}\right] = \delta_{\vec{k}\vec{k}'}\delta_{\sigma\sigma'} \tag{7}$$

将电磁场 Hamiltonion 转入粒子数表象

$$\hat{H} = \int \left[\frac{\vec{P}^2}{2\varepsilon_0} + \frac{1}{2\mu_0}\left(\nabla \times \vec{A}\right)^2\right] \mathrm{d}\tau = \sum_{\vec{k},\sigma} \hbar\omega_{\vec{k}} \left(\hat{C}^\dagger_{\vec{k},\sigma}\hat{C}_{\vec{k},\sigma} + \frac{1}{2}\right) \tag{8}$$

1.1.2 方法二：仿谐振子方法

我们在初等量子力学中已经学习过一维线性谐振子的量子化

$$\begin{cases} \hat{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^\dagger\right) \\ \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right) \end{cases} \tag{9}$$

其中湮灭和产生算符表示为

$$\begin{cases} \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} + i\frac{1}{\sqrt{2m\hbar\omega}}\hat{p} \\ \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{q} - i\frac{1}{\sqrt{2m\hbar\omega}}\hat{p} \end{cases} \tag{10}$$

其 Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2 = \frac{m}{2}\left(\omega^2\hat{q}^2 + \dot{\hat{q}}^2\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \tag{11}$$

仿照线性谐振子量子化方法，将电磁场进行驻波模式的拆解

$$E_y = AG(t)\sin(kx) \qquad H_z(x,t) = A\frac{\varepsilon_0}{k}\dot{G}(t)\cos(kx) \tag{12}$$

其中 $k = \pi n/a$，$n = 1, 2, 3, \cdots$。电磁场 Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}\int \left(\varepsilon_0 E_y^2 + \mu_0 H_z^2\right) \mathrm{d}\tau \\ &= \frac{1}{2}\int \left[\varepsilon_0 A^2 G^2(t)\sin^2(kx) + \mu_0 A^2 \frac{\varepsilon_0^2}{k^2}\dot{G}^2(t)\cos^2(kx)\right] \mathrm{d}\tau \\ &= \frac{1}{2}\varepsilon_0 A^2 G^2(t) \iint \mathrm{d}y\mathrm{d}z \int_0^a \sin^2(kx)\mathrm{d}x + \frac{1}{2}\mu_0 A^2 \frac{\varepsilon_0^2}{k^2}\dot{G}^2(t) \iint \mathrm{d}y\mathrm{d}z \int_0^a \cos^2(kx)\mathrm{d}x \\ &= \frac{1}{4}\varepsilon_0 V A^2 \left[G^2(t) + \frac{\mu_0\varepsilon_0}{k^2}\dot{G}^2(t)\right] = \frac{1}{2}\frac{\varepsilon_0 V A^2}{2\omega^2}\left[\omega^2 G^2(t) + \dot{G}^2(t)\right] \end{aligned} \tag{13}$$

和线性谐振子进行类比

$$m \leftrightarrow \frac{\varepsilon_0 V A^2}{2\omega^2} \qquad q \leftrightarrow G(t) \qquad p = m\dot{q} \leftrightarrow m\dot{G}(t)$$

故

$$\hat{G}(t) = \frac{1}{A}\sqrt{\frac{\hbar\omega}{\varepsilon_0 V}}\left(\hat{a} + \hat{a}^\dagger\right) \tag{14}$$

$$\dot{\hat{G}}(t) = -\frac{i}{A}\sqrt{\frac{\hbar\omega^3}{\varepsilon_0 V}}\left(\hat{a} - \hat{a}^\dagger\right) \tag{15}$$

1.2 数态或 Fock 态

设本征态 $|n\rangle$ 对应的本征值为 E_n，有

$$H|n\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|n\rangle = E_n|n\rangle \tag{16}$$

利用波色子对易关系

$$[a, a^\dagger] = 1 \qquad\qquad [a, a] = [a^\dagger, a^\dagger] = 0$$

有

$$Ha|n\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)a|n\rangle = \hbar\omega\left(aa^\dagger - \frac{1}{2}\right)a|n\rangle = \hbar\omega a\left(a^\dagger a - \frac{1}{2}\right)|n\rangle = (E_n - \hbar\omega)a|n\rangle \tag{17}$$

$a|n\rangle$ 对应 $n-1$ 能级，这意味着

$$|n-1\rangle = \frac{a}{\alpha_n}|n\rangle \tag{18}$$

重复 n 次此过程，

$$Ha|0\rangle = (E_0 - \hbar\omega)\,a|0\rangle \tag{19}$$

E_0 为基态能量，而 $E_0 - \hbar\omega < E_0$，这明显不符合我们的要求。因此为了使等式成立，一定有

$$a|0\rangle = 0 \tag{20}$$

$|0\rangle$ 对应于真空态

$$H|0\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega|0\rangle = E_0|0\rangle \tag{21}$$

得到基态能量 $E_0 = \frac{1}{2}\hbar\omega$，故

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{22}$$

因此，由

$$H|n\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|n\rangle = \left(n + \frac{1}{2}\right)\hbar\omega|n\rangle \tag{23}$$

得到

$$a^\dagger a|n\rangle = n|n\rangle \tag{24}$$

能量本征态 $|n\rangle$ 同样是光子数算符 $n = a^\dagger a$ 的本征态。接下来我们来确定系数 $|\alpha_n|$，

$$\langle n-1|n-1\rangle = \frac{1}{|\alpha_n|^2}\langle n|a^\dagger a|n\rangle = \frac{n}{|\alpha_n|^2}\langle n|n\rangle = \frac{n}{|\alpha_n|^2} = 1 \tag{25}$$

得到 $\alpha_n = \sqrt{n}$，于是

$$a|n\rangle = \sqrt{n}\,|n-1\rangle \tag{26}$$

用产生算符 $a^\dagger$ 重复此过程，得到

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{27}$$

故

$$|n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle \tag{28}$$

$|n\rangle$ 被称为数态 (number state)，也被称为 Fock 态。

```
# Fock态
fock(N, n)  # Hilbert空间维数N , 光子数n
basis(N, n)
# 湮灭算符与产生算符
destroy(N)
destroy(N).dag()
# 光子数算符
num(N)
```

1.3 真空涨落

Fock 态 $|n\rangle$ 有一个重要性质。单模线性偏振场算符

$$\hat{E}(\vec{r},t)=\mathcal{E}\hat{a}e^{-i\omega t+i\vec{k}\cdot\vec{r}}+\text{H.C.} \tag{29}$$

关于 fock 态的期望值为 0，即

$$\langle n|\,\hat{E}\,|n\rangle=0 \tag{30}$$

然而，强度算符 $\hat{E}^2$ 的期待值不为 0，即

$$\langle n|\,\hat{E}^2\,|n\rangle=2|\mathcal{E}|^2\left(n+\frac{1}{2}\right) \tag{31}$$

$$\left\langle(\Delta\hat{E})^2\right\rangle=\langle E^2\rangle-\langle E\rangle^2=2|\mathcal{E}|^2\left(n+\frac{1}{2}\right) \tag{32}$$

值得注意的是，当系统处于真空态 $|0\rangle$ 时，仍然存在真空涨落

$$\langle 0|\,(\Delta\hat{E})^2\,|0\rangle=|\mathcal{E}|^2=\frac{\hbar\omega}{2\varepsilon_0 V} \tag{33}$$

2 电磁场的相干态 (coherent state) 和压缩态 (squeezed state)

2.1 相干态

- 相干态是光子湮灭算符的本征态
- 相干态是平移的真空态
- 相干态是符合最小测不准关系的量子态

2.1.1 光子湮灭算符的本征态

相干态 $|\alpha\rangle$ 是光子湮灭算符 a 的本征态，即有

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{34}$$

将 $|\alpha\rangle$ 按 fock 态展开

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle \tag{35}$$

$$a|\alpha\rangle = \sum_{n=0}^{\infty} c_n a |n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}|n\rangle = \sum_{n=0}^{\infty} \alpha c_n |n\rangle \tag{36}$$

可以得到

$$c_{n+1} = \alpha \frac{c_n}{\sqrt{n+1}} \qquad \Rightarrow \qquad c_n = \alpha^n \frac{c_0}{\sqrt{n!}} \tag{37}$$

于是

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{38}$$

归一化

$$\langle\alpha|\alpha\rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} = 1 \tag{39}$$

得到

$$c_0 = e^{-|\alpha|^2/2} \tag{40}$$

故

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{41}$$

又

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \tag{42}$$

故

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} |0\rangle \tag{43}$$

根据 Taylor 展开 $e^x = \sum_{n=0}^{\infty} x^n/n!$，有

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle \tag{44}$$

根据谱映射定理，有

$$a|0\rangle = 0|0\rangle \qquad e^{-\alpha^* a}|0\rangle = e^{-\alpha^*\times 0}|0\rangle = |0\rangle \tag{45}$$

于是

$$|\alpha\rangle = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a}|0\rangle = D(\alpha)|0\rangle \tag{46}$$

其中 $D(\alpha)$ 为**位移算符**

$$D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a} \tag{47}$$

根据 Baker-Hausdorff 公式，若任意两个算符 A, B 满足 $[[A,B],A] = [[A,B],B] = 0$，则有

$$e^{A+B} = e^{-[A,B]/2}e^A e^B \tag{48}$$

令 $A = \alpha a^\dagger$，$B = -\alpha^* a$，

$$e^{\alpha a^\dagger - \alpha^* a} = e^{-[\alpha a^\dagger, -\alpha^* a]/2}e^{\alpha a^\dagger}e^{-\alpha^* a} = e^{\alpha\alpha^*[a^\dagger, a]/2}e^{\alpha a^\dagger}e^{-\alpha^* a} = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a} \tag{49}$$

故

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{50}$$

位移算符 $D(\alpha)$ 是一个幺正算符

$$D^\dagger(\alpha) = D(-\alpha) = [D(\alpha)]^{-1} \tag{51}$$

对于任意算符 A, B，有

$$e^{-\alpha A}Be^{\alpha A} = B - \alpha[A,B] + \frac{\alpha^2}{2!}[A,[A,B]] + \cdots \tag{52}$$

令 $A = a^\dagger$，$B = a$，则

$$e^{-\alpha a^\dagger}ae^{\alpha a^\dagger} = a + \alpha \tag{53}$$

于是

$$D^\dagger(\alpha)aD(\alpha) = e^{\alpha^* a}e^{-\alpha a^\dagger}ae^{\alpha a^\dagger}e^{-\alpha^* a} = e^{\alpha^* a}(a+\alpha)e^{-\alpha^* a} = a + \alpha \tag{54}$$

同理

$$D^\dagger(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^* \tag{55}$$

故

$$D^\dagger(\alpha)f(a, a^\dagger)D(\alpha) = f(a+\alpha, a^\dagger + \alpha^*) \tag{56}$$

相干态由位移算符和真空态表示

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{57}$$

```
# QuTiP
coherent(N, alpha) # 相干态
displace(N, alpha)  # 位移算符
```

2.1.2 相干态在坐标表象的波函数

相干态在坐标表象的波函数

$$\psi_\alpha(x) = \langle x|\alpha\rangle \tag{58}$$

由第一章可知

$$a = \left(\frac{\mu\omega}{2\hbar}\right)^{1/2}\left(x + \frac{i}{\mu\omega}p_x\right) = \left(\frac{\mu\omega}{2\hbar}\right)^{1/2}\left(x + \frac{\hbar}{\mu\omega}\frac{\partial}{\partial x}\right) \tag{59}$$

则

$$\langle x|\, a\, |x'\rangle = \left(\frac{\mu\omega}{2\hbar}\right)^{1/2}\left(x + \frac{\hbar}{\mu\omega}\frac{\partial}{\partial x}\right)\delta(x - x') \tag{60}$$

令 $\xi = \sqrt{\frac{\mu\omega}{\hbar}}x$，进行无量纲化处理，得

$$\langle x|\, a\, |x'\rangle = \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial \xi}\right)\delta\left(x - x'\right) \tag{61}$$

由于

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle \qquad \langle x|\, a\,|\alpha\rangle = \alpha\,\langle x|\alpha\rangle \tag{62}$$

又

$$\langle x|\, a\,|\alpha\rangle = \int \langle x|\, a\,|x'\rangle\,\langle x'|\alpha\rangle\,\mathrm{d}x' = \int \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial \xi}\right)\delta(x - x')\,\langle x'|\alpha\rangle\,\mathrm{d}x' = \frac{1}{\sqrt{2}}\left(\xi + \frac{\partial}{\partial \xi}\right)\langle x|\alpha\rangle = \alpha\,\langle x|\alpha\rangle \tag{63}$$

令 $\langle x|\alpha\rangle = f$，则

$$\frac{\partial f}{\partial \xi} = -(\xi - \sqrt{2}\alpha)f \tag{64}$$

解出

$$f = \langle x|\alpha\rangle = Ae^{-\frac{1}{2}(\xi - \sqrt{2}\alpha)^2} \tag{65}$$

归一化求得系数 A，系数 A 可相差一个与 x 无关的相位因子 δ

$$A = \left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4} e^{\frac{1}{4}(\alpha - \alpha^*)^2} e^{i\delta} \tag{66}$$

于是

$$\psi_\alpha(x) = \langle x|\alpha\rangle = \left(\frac{\mu\omega}{\pi\hbar}\right)^{1/4} e^{\frac{1}{4}(\alpha - \alpha^*)^2} e^{i\delta} e^{-\frac{1}{2}(\xi - \sqrt{2}\alpha)^2} \tag{67}$$

2.1.3 相干态中电场的涨落

行波情况，

$$\hat{E}(\vec{r}, t) = i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\hat{a}e^{-i\omega t + i\vec{k}\cdot\vec{r}} + \text{H.C.} \tag{68}$$

电场能量的期待值

$$\begin{aligned}
\langle\alpha|\,\hat{E}(\vec{r}, t)\,|\alpha\rangle =& i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\langle\alpha|\,\hat{a}\,|\alpha\rangle\, e^{-i\omega t + i\vec{k}\cdot\vec{r}} - \langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle\, e^{i\omega t - i\vec{k}\cdot\vec{r}}\right) \\
=& i\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\left(\alpha e^{-i\omega t + i\vec{k}\cdot\vec{r}} - \alpha^* e^{i\omega t - i\vec{k}\cdot\vec{r}}\right) \\
=& 2\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}|\alpha|\cos\left(\omega t - \vec{k}\cdot\vec{r} + \theta\right) \qquad \alpha = |\alpha|e^{-i\theta}
\end{aligned} \tag{69}$$

$$\begin{aligned}\langle\alpha|\hat{E}^2(\vec{r},t)|\alpha\rangle &= -\frac{\hbar\omega}{2\varepsilon_0 V}\Big(\langle\alpha|\hat{a}^2|\alpha\rangle e^{-2i\omega t+2i\vec{k}\cdot\vec{r}} + \langle\alpha|\hat{a}^{\dagger 2}|\alpha\rangle e^{2i\omega t-2i\vec{k}\cdot\vec{r}} - 2\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle - 1\Big) \\ &= -\frac{\hbar\omega}{2\varepsilon_0 V}\Big(\alpha^2 e^{-2i\omega t+2i\vec{k}\cdot\vec{r}} + \alpha^{*2}e^{2i\omega t-2i\vec{k}\cdot\vec{r}} - 2|\alpha|^2 - 1\Big)\end{aligned} \tag{70}$$

电场能量涨落

$$\Big\langle(\Delta\hat{E})^2\Big\rangle = \langle\alpha|\hat{E}^2|\alpha\rangle - (\langle\alpha|\hat{E}|\alpha\rangle)^2 = \frac{\hbar\omega}{2\varepsilon_0 V} \tag{71}$$

2.1.4 相干态是最接近经典的量子态

量子化单模光场相当于一个量子化谐振子，我们在 Schrödinger 绘景下研究振动坐标在相干态光场的平均值随时间变化规律。设初态为相干态 $|\psi(t=0)\rangle = |\alpha\rangle$，系统 Hamiltonian

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) \tag{72}$$

任意时刻系统态矢量

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(t=0)\rangle = e^{-i\omega t(a^\dagger a+\frac{1}{2})}|\alpha\rangle \tag{73}$$

- **方法一**

$$\begin{aligned}|\psi(t)\rangle =& e^{-i\omega t(a^\dagger a+\frac{1}{2})}|\alpha\rangle = e^{-i\omega t(a^\dagger a+\frac{1}{2})}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\ =& e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega t(a^\dagger a+\frac{1}{2})}|n\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega t(n+\frac{1}{2})}|n\rangle \\ =& e^{-i\omega t/2}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega tn}|n\rangle = e^{-i\omega t/2}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}}|n\rangle \\ =& e^{-i\omega t/2}e^{-|\alpha e^{-i\omega t}|^2/2}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}}|n\rangle = e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle\end{aligned} \tag{74}$$

- **方法二**：令 $\theta = \omega t$，则

$$|\psi(t)\rangle = e^{-i\omega t(a^\dagger a+\frac{1}{2})}|\alpha\rangle = e^{-i\theta/2}e^{-i\theta a^\dagger a}|\alpha\rangle \tag{75}$$

  设转动算符

$$R(\theta) = e^{-i\theta a^\dagger a} \tag{76}$$

  将转动算符作用在位移算符上

$$\begin{aligned}R(\theta)D(\alpha)R^{-1}(\theta) &= \exp\big(-i\theta a^\dagger a\big)\exp\big(\alpha a^\dagger - \alpha^* a\big)\exp\big(i\theta a^\dagger a\big) \\ &= \exp\big[\exp\big(-i\theta a^\dagger a\big)\alpha a^\dagger\exp\big(i\theta a^\dagger a\big) - \exp\big(-i\theta a^\dagger a\big)\alpha^* a\exp\big(i\theta a^\dagger a\big)\big] \\ &= \exp\big(\alpha e^{-i\theta}a^\dagger - \alpha^* e^{i\theta}a\big) = D(\alpha e^{-i\theta})\end{aligned} \tag{77}$$

  即在转动变换下位移算符保持不变，仅仅使位移量多了一个相位因子。

$$R(\theta)|\alpha\rangle = R(\theta)D(\alpha)|0\rangle = R(\theta)D(\alpha)R^{-1}(\theta)R(\theta)|0\rangle = D(\alpha e^{-i\theta})|0\rangle = \left|\alpha e^{-i\theta}\right\rangle \tag{78}$$

  于是

$$|\psi\rangle = e^{-i\theta/2}\left|\alpha e^{-i\theta}\right\rangle = e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle \tag{79}$$

振动坐标在相干态光场的平均值

$$\begin{aligned}\langle\psi| x |\psi\rangle &= \left\langle \alpha e^{-i\omega t}\right| \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) \left|\alpha e^{-i\omega t}\right\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}) \\ &= \sqrt{\frac{\hbar}{2m\omega}}|\alpha| \left[e^{-i(\omega t+\theta)} + e^{i(\omega t+\theta)}\right] = \sqrt{\frac{2\hbar}{m\omega}}|\alpha| \cos(\omega+\theta)\end{aligned} \tag{80}$$

```
# Time evolution and quantum system dynamics
# Schrödinger equation solver provided for the unitary Schrödinger equation
result = sesolve(H, psi0, e_ops=[], args={})
# Master equation solver provided for the Lindblad master equation and von Neumann equation
result = mesolve(H, rho0, tlist, c_ops=[], e_ops=[], args={})
# Monte Carlo solver
result = mcsolve(H, psi0, tlist, c_ops=[], e_ops=[], ntraj=0, args={})

result.times # List/array of times at which simulation data is calculated
result.expect # List/array of expectation values
result.states # List/array of state vectors/density matrices calculated at times
```

```
from qutip import *
import numpy as np
import matplotlib.pyplot as plt

# parameters
N = 10
w = 2 * np.pi
a = destroy(N)
times = np.linspace(0, 10, 200)

psi0 = coherent(N, 0.5-0.5j)
H = w * (a.dag()*a + 0.5) # Hamiltonian

# Calculate
result = mesolve(H, psi0, times, [], [(a+a.dag())/np.sqrt(2*omega)])

plt.plot(times, result.expect[0])
plt.xlabel('Time')
plt.ylabel('Expectation Values')
plt.show()
```

2.1.5 线性谐振子相干态下的测不准关系

由对易关系 $[x,p]=i\hbar$，存在测不准关系

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{81}$$

线性谐振子坐标与动量可用产生与湮灭算符表示

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) \qquad p = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$$

在相干态 $|\alpha(t)\rangle$ 下，

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \tag{82}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{m\hbar\omega}{2}} \tag{83}$$

于是

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{84}$$

故相干态是 x 和 p 的最小不确定态。

定义两个幅度算符

$$X_1 = \frac{1}{2}(a + a^\dagger) \qquad\qquad X_2 = \frac{1}{2i}(a - a^\dagger)$$

由

$$[X_1, X_2] = \frac{i}{2} \tag{85}$$

有

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4} \tag{86}$$

计算 X_1 和 X_2 的均方根偏差 (rms)

$$\langle X_1 \rangle = \frac{1}{2} \langle \alpha | (a + a^\dagger) | \alpha \rangle = \frac{1}{2}(\alpha + \alpha^*) \tag{87}$$

$$\langle X_1^2 \rangle = \frac{1}{4} \langle \alpha | (a + a^\dagger)^2 | \alpha \rangle = \frac{1}{4} \langle \alpha | (aa + a^\dagger a^\dagger + 2a^\dagger a + 1) | \alpha \rangle = \frac{1}{4}(\alpha + \alpha^*)^2 + \frac{1}{4} \tag{88}$$

$$\Delta X_1 = \sqrt{\langle X_1^2 \rangle - \langle X_1 \rangle^2} = \frac{1}{2} \tag{89}$$

同理

$$\Delta X_2 = \sqrt{\langle X_2^2 \rangle - \langle X_2 \rangle^2} = \frac{1}{2} \tag{90}$$

即标准量子噪声极限。

2.1.6 相干态的性质

1. **光子数不确定，且光子分布为 Poisson 分布**。相干态

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{91}$$

相干态中出现 n 个光子的概率

$$P_n = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \tag{92}$$

平均光子数

$$\langle n \rangle = \langle \alpha | a^\dagger a | \alpha \rangle = |\alpha|^2 \tag{93}$$

$$\langle n^2 \rangle = \langle \alpha | a^\dagger a a^\dagger a | \alpha \rangle = \langle \alpha | a^\dagger (a^\dagger a + 1) a | \alpha \rangle = \langle \alpha | \left[(a^\dagger)^2 a^2 + a^\dagger a \right] | \alpha \rangle = |\alpha|^4 + |\alpha|^2 \tag{94}$$

均方偏差

$$\langle (\Delta n)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 = |\alpha|^2 \tag{95}$$

得到 Poisson 分布

$$P_n = e^{-\langle n \rangle} \frac{\langle n \rangle^n}{n!} \tag{96}$$

2. **相干态是最小不确定态（最接近经典态的量子态）**

$$\Delta x \Delta p = \frac{\hbar}{2} \tag{97}$$

令 $X_1 = (a + a^\dagger)/2$，$X_2 = (a - a^\dagger)/2i$，则 $[X_!, X_2] = i/2$，得到 $\Delta X_1 \Delta X_2 \geq 1/4$。在相干态时，有

$$\Delta X_1 = \Delta X_2 = \frac{1}{2} \qquad \Delta X_1 \Delta X_2 = \frac{1}{4} \tag{98}$$

3. **相干态构成（超）完全集**

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, \mathrm{d}^2\alpha = 1 \tag{99}$$

对整个复平面进行积分，$\alpha = x + iy = re^{i\theta}$，

$$\mathrm{d}^2\alpha = \mathrm{d}[\mathrm{Re}(\alpha)]\mathrm{d}[\mathrm{Im}(\alpha)] = \mathrm{d}x\mathrm{d}y = r\mathrm{d}r\mathrm{d}\theta \tag{100}$$

于是

$$\begin{aligned}
\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|\, \mathrm{d}^2\alpha &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\pi}\frac{|n\rangle\langle m|}{\sqrt{n!m!}}\int_0^\infty e^{-|\alpha|^2}\alpha^n(\alpha^*)^m \mathrm{d}^2\alpha \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\pi}\frac{|n\rangle\langle m|}{\sqrt{n!m!}}\int_0^\infty e^{-r^2}r^{n+m+1}\mathrm{d}r\int_0^{2\pi}e^{i(n-m)\theta}\mathrm{d}\theta \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{2\pi}\frac{|n\rangle\langle m|}{\sqrt{n!m!}}\int_0^\infty e^{-r^2}r^{n+m}\mathrm{d}r^2(2\pi\delta_{nm}) \\
&= \sum_{n=0}^{\infty}\frac{|n\rangle\langle n|}{n!}\int_0^\infty e^{-r^2}r^{2n}\mathrm{d}r^2 = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1
\end{aligned} \tag{101}$$

4. **不同相干态不正交，但仍可归一化**

$$\langle\alpha| = e^{-|\alpha|^2/2}\sum_{m=0}^{\infty}\frac{(\alpha^*)^m}{\sqrt{m!}}\langle m| \qquad |\beta\rangle = e^{-|\beta|^2/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle \tag{102}$$

$$\langle\alpha|\beta\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right) \tag{103}$$

$$|\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \tag{104}$$

$|\alpha - \beta|$ 的值量度了复平面上相干态 $|\alpha\rangle$ 和 $|\beta\rangle$ 偏离正交性的程度。

2.2 压缩态

2.2.1 压缩态的物理起源

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0x = \frac{p^2}{2m} + \frac{1}{2}k\left(x - \frac{eE_0}{k}\right)^2 - \frac{1}{2}k\left(\frac{eE_0}{k}\right)^2 \tag{105}$$

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0(ax - bx^2) = \frac{p^2}{2m} + \frac{1}{2}\left(k + 2ebE_0\right)x^2 - eaE_0x \tag{106}$$

2.2.2 定义

设 $[A,B]=iC$，有 $\Delta A\Delta B\geq |\langle C\rangle|/2$，关于 A,B 的最小不确定态满足

$$\Delta A\Delta B=\frac{|\langle C\rangle|}{2} \tag{107}$$

值得注意的是，最小不确定态与考虑的算符组有关。例如相干态是关于 X_1,X_2 的最小不确定态，而不是关于 $a,a^\dagger$ 的最小不确定态。

通过不确定关系定义**压缩态**。若满足

$$\Delta A<\sqrt{\frac{|\langle C\rangle|}{2}}\qquad\qquad \Delta B<\sqrt{\frac{|\langle C\rangle|}{2}} \tag{108}$$

则关于 A,B 的态被称为压缩态。

2.2.3 常用压缩态和不确定关系

压缩态与考虑的算符组有关，我们关心的是电场的相位正交 (quadrature) 的两幅度算符

$$X_1=\frac{1}{2}(a+a^\dagger)\qquad\qquad X_2=\frac{1}{2i}(a-a^\dagger)$$

根据不确定关系

$$\Delta X_1\Delta X_2\geq\frac{1}{4} \tag{109}$$

电场压缩态满足

$$\Delta X_i<\frac{1}{2}\qquad (i=1\text{ or }2) \tag{110}$$

若满足 Eq.(110) 的同时满足

$$\Delta X_1\Delta X_2=\frac{1}{4} \tag{111}$$

则称其为理想压缩态。

2.2.4 压缩算符和压缩相干态

$$S(\xi)=\exp\left(\frac{1}{2}\xi^* a^2-\frac{1}{2}\xi a^{\dagger 2}\right) \tag{112}$$

其中 $\xi=re^{i\theta}$ 为任意复数，显而易见 $S^\dagger(\xi)=S^{-1}(\xi)=S(-\xi)$。利用公式

$$e^ABe^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots \tag{113}$$

可以得到

$$S^\dagger(\xi)aS(\xi)=a\cosh r-a^\dagger e^{i\theta}\sinh r \tag{114}$$

$$S^\dagger(\xi)a^\dagger S(\xi)=a^\dagger\cosh r-ae^{-i\theta}\sinh r \tag{115}$$

设

$$Y_1+iY_2=(X_1+iX_2)e^{-i\theta/2}=ae^{-i\theta/2} \tag{116}$$

于是

$$S^\dagger(\xi)(Y_1+iY_2)S(\xi)=Y_1e^{-r}+iY_2e^r \tag{117}$$

- **压缩相干态**

$$|\alpha,\xi\rangle = S(\xi)D(\alpha)|0\rangle \tag{118}$$

- **相干压缩态**

$$|\xi,\alpha\rangle = D(\alpha)S(\xi)|0\rangle \tag{119}$$

- **真空压缩态**

$$|\xi\rangle = |\xi,\alpha=0\rangle = S(\xi)|0\rangle \tag{120}$$

将真空压缩态按 Fock 态展开 $|\xi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$。由于

$$S(\xi)aS^\dagger(\xi)|\xi\rangle = S(\xi)aS^\dagger(\xi)S(\xi)|0\rangle = S(\xi)a|0\rangle = 0 \tag{121}$$

其中

$$S(\xi)aS^\dagger(\xi) = a\cosh r + a^\dagger e^{i\theta}\sinh r = \mu a + \gamma a^\dagger \tag{122}$$

将 Eq.(121) 用 Fock 展开

$$S(\xi)aS^\dagger(\xi)|\xi\rangle = \sum_{n=0}^{\infty} c_n(\mu a + \gamma a^\dagger)|n\rangle = \sum_{n=0}^{\infty} c_n(\mu\sqrt{n}|n-1\rangle + \gamma\sqrt{n+1}|n+1\rangle) = 0 \tag{123}$$

将 $\langle m|$ 作用于上式

$$\begin{aligned}\langle m|(\mu a + \gamma a^\dagger)|\xi\rangle &= \sum_{n=0}^{\infty} c_n(\mu\sqrt{n}\langle m|n-1\rangle + \gamma\sqrt{n+1}\langle m|n+1\rangle) \\ &= \begin{cases} c_1\mu = 0 & m=0 \\ c_{m+1}\mu\sqrt{m+1} + c_{m-1}\gamma\sqrt{m} = 0 & m \ge 1 \end{cases}\end{aligned} \tag{124}$$

即

$$c_{n+1} = -\frac{\gamma}{\mu}\sqrt{\frac{n}{n+1}}c_{n-1} \tag{125}$$

通过迭代得到

$$\begin{cases} c_{2n} = (-1)^n\sqrt{\frac{(2n-1)!!}{(2n)!!}}e^{in\theta}\tanh^n r c_0 = (-1)^n\frac{\sqrt{(2n)!!}}{2^n n!}e^{in\theta}\tanh^n r c_0 \\ c_{2n+1} = 0 \end{cases} \tag{126}$$

通过归一化确定系数 c_0

$$\langle\xi|\xi\rangle = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_m^* c_n\langle m|n\rangle = \sum_{n=0}^{\infty}|c_n|^2 = c_0^2\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\tanh^{2n} r = c_0^2\cosh r = 1 \tag{127}$$

即

$$c_0 = \frac{1}{\sqrt{\cosh r}} \tag{128}$$

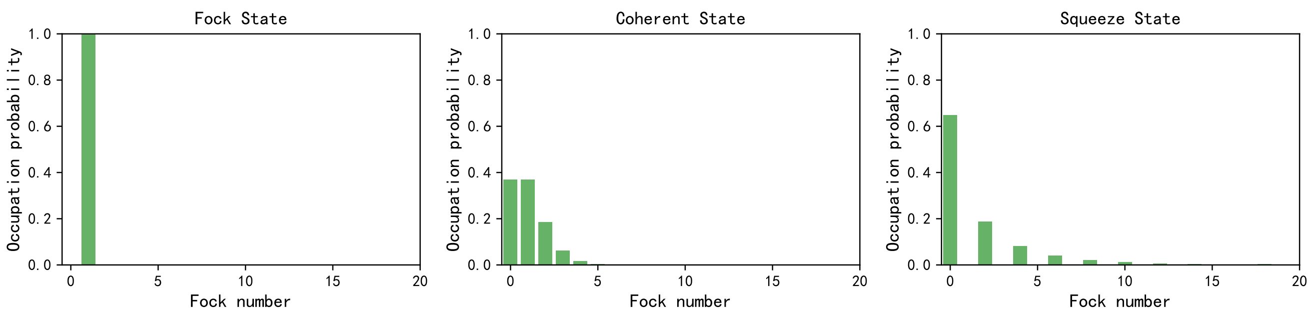
故

$$|\xi\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}(-1)^n\frac{\sqrt{(2n)!}}{2^n n!}e^{in\theta}\tanh^n r|2n\rangle \tag{129}$$

```
squeeze(N, sp)     # Squeezing operator (Single-mode)
squeezing(q1, q2, sp)     # Squeezing operator (Generalized)
```

```
N = 20
rho_fock = fock_dm(N, 1)
rho_coherent = coherent_dm(N, 1)
rho_squeeze = squeeze(N, 1)*fock(N,0)*fock(N,0).dag()*squeeze(N, 1).dag()

fig, axes = plt.subplots(1, 3, figsize=(12,3))
plot_fock_distribution(rho_fock, fig=fig, ax=axes[0], title='Fock State')
plot_fock_distribution(rho_coherent, fig=fig, ax=axes[1], title='Coherent State')
plot_fock_distribution(rho_squeeze, fig=fig, ax=axes[2], title='Squeeze State')
fig.tight_layout()
plt.show()
```



2.2.5 常用的平均值和方差

$$\begin{aligned}\langle a\rangle = &\langle\alpha,\xi|\, a\,|\alpha,\xi\rangle = \langle 0|\, D^\dagger(\alpha)S^\dagger(\xi)aS(\xi)D(\alpha)\,|0\rangle \\ = &\langle\alpha|\,(a\cosh r - a^\dagger e^{i\theta}\sinh r)\,|\alpha\rangle \\ = &\alpha\cosh r - \alpha^* e^{i\theta}\sinh r\end{aligned} \tag{130}$$

$$\begin{aligned}\langle a^2\rangle = &\langle (a^\dagger)^2\rangle = \langle 0|\, D^\dagger(\alpha)S^\dagger(\xi)a^2S(\xi)D(\alpha)\,|0\rangle \\ = &\langle\alpha|\, S^\dagger(\xi)aS(\xi)S^\dagger(\xi)aS(\xi)\,|\alpha\rangle = \langle\alpha|\,(a\cosh r - a^\dagger e^{i\theta}\sinh r)^2\,|\alpha\rangle \\ = &\alpha^2\cosh^2 r + (\alpha^*)^2 e^{2i\theta}\sinh^2 r - 2|\alpha|^2 e^{i\theta}\sinh r\cosh r - e^{i\theta}\cosh r\sinh r\end{aligned} \tag{131}$$

$$\langle a^\dagger a\rangle = |\alpha|^2(\cosh^2 r + \sinh^2 r) - (\alpha^*)^2 e^{i\theta}\sinh r\cosh r - \alpha^2 e^{-i\theta}\sinh r\cosh r + \sinh^2 r \tag{132}$$

由 $Y_1 + iY_2 = a\exp(-i\theta/2)$，计算得到

$$\Delta Y_1 = \sqrt{\langle Y_1^2\rangle - \langle Y_1\rangle^2} = \frac{1}{4}e^{-2r} \tag{133}$$

$$\Delta Y_2 = \sqrt{\langle Y_2^2\rangle - \langle Y_2\rangle^2} = \frac{1}{4}e^{2r} \tag{134}$$

$$\Delta Y_1 \Delta Y_2 = \frac{1}{4} \tag{135}$$

3 量子分布理论和部分相干辐射

3.1 密度算符

3.1.1 纯态

系统处于纯态时，其状态可用态矢量 $|\psi\rangle = \sum_n c_n |n\rangle$ 表示。设 $A|n\rangle = a_n |n\rangle$，则力学量 A 的平均值

$$\langle A\rangle = \langle\psi| A |\psi\rangle = \sum_n |c_n|^2 a_n \tag{136}$$

定义求迹运算 $\mathrm{tr}(A) = \sum_n \langle n| A |n\rangle$，则有

$$\mathrm{tr}(|\psi\rangle\langle\phi|) = \sum_n \langle n|\psi\rangle\langle\phi|n\rangle = \sum_n \langle\phi|n\rangle\langle n|\psi\rangle = \langle\phi|\left(\sum_n |n\rangle\langle n|\right)|\psi\rangle = \langle\phi|\psi\rangle \tag{137}$$

故

$$\langle A\rangle = \langle\psi|(A|\psi\rangle) = \mathrm{tr}[(A|\psi\rangle)\langle\psi|] = \mathrm{tr}(A|\psi\rangle\langle\psi|) \tag{138}$$

定义**密度算符**

$$\rho = |\psi\rangle\langle\psi| \tag{139}$$

则有

$$\langle A\rangle = \langle\psi| A |\psi\rangle = \mathrm{tr}(\rho A) \tag{140}$$

纯态下，密度算符具有如下性质

1. $\mathrm{tr}(\rho) = 1$
$$\mathrm{tr}(\rho) = \sum_n \langle n|\rho|n\rangle = \sum_n \langle n|\psi\rangle\langle\psi|n\rangle = \sum_n \langle\psi|n\rangle\langle n|\psi\rangle = \langle\psi|\psi\rangle = 1 \tag{141}$$
2. $\rho^\dagger = \rho$
$$\rho^\dagger = (|\psi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\psi| = \rho \tag{142}$$
3. $\rho^2 = \rho$
$$\rho^2 = (|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho \tag{143}$$
4. $\mathrm{tr}(\rho^2) = 1$

3.1.2 混态

由于统计物理或量子力学本身的原因，系统状态有时无法用一个态矢量来表示，系统不处在一个确定的态中，而是以 $p_1, p_2, \cdots$ 的几率处于态 $|\psi_1\rangle, |\psi_2\rangle, \cdots$ 中，此时我们称系统状态为**混态**。混态的平均值有两层含义：(1) 量子力学平均，即对每个分态求平均；(2) 统计平均，即对各分态在系综内出现的概率求平均。即

$$\begin{aligned}\langle\bar{A}\rangle &= \sum_j p_j \langle\psi_j| A |\psi_j\rangle = \sum_n\sum_j p_j \langle\psi_j|n\rangle\langle n| A |\psi_j\rangle \\ &= \sum_n\sum_j \langle n| A p_j |\psi_j\rangle\langle\psi_j|n\rangle = \sum_n \langle n| A \sum_j (p_j |\psi_j\rangle\langle\psi_j|)|n\rangle \\ &= \sum_n \langle n| A\rho |n\rangle = \mathrm{tr}(\rho A)\end{aligned} \tag{144}$$

密度算符

$$\rho = \sum_j p_j |\psi_j\rangle \langle\psi_j| \tag{145}$$

混态下密度算符具有如下性质

1. $\mathrm{tr}(\rho) = \sum_j p_j = 1$
2. $\rho^\dagger = \rho$
3. $\mathrm{tr}(\rho^2) < \mathrm{tr}(\rho) = 1$

$$\begin{aligned}\rho^2 &= \sum_j p_j |\psi_j\rangle \langle\psi_j| \sum_i p_i |\psi_i\rangle \langle\psi_i| = \sum_{i,j} p_i p_j |\psi_i\rangle \langle\psi_i|\psi_j\rangle \langle\psi_j| \\ &= \sum_{i,j} p_i p_j \delta_{ij} |\psi_i\rangle \langle\psi_j| = \sum_j p_j^2 |\psi_j\rangle \langle\psi_j|\end{aligned} \tag{146}$$

$$\mathrm{tr}\left(\rho^2\right) = \sum_j p_j^2 \langle\psi_j|\psi_j\rangle = \sum_j p_j^2 < \sum_j p_j = \mathrm{tr}(\rho) = 1 \tag{147}$$

3.1.3 Master 方程

- 态矢量随时间演化方程：Schrödinger 方程

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = \hat{H} |\psi\rangle \tag{148}$$

- 力学量随时间演化方程：Heisenberg 方程

$$i\hbar \frac{\partial \hat{A}}{\partial t} = [\hat{A}, \hat{H}] \tag{149}$$

- 密度算符随时间演化方程：Master 方程

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \tag{150}$$

从 Schrödinger 方程到 Master 方程

$$\begin{aligned}i\hbar \frac{\partial \hat{\rho}}{\partial t} =& i\hbar \frac{\partial (|\psi\rangle \langle\psi|)}{\partial t} = i\hbar \left(\frac{\partial |\psi\rangle}{\partial t}\right) \langle\psi| + i\hbar |\psi\rangle \frac{\partial \langle\psi|}{\partial t} \\ =& \hat{H} |\psi\rangle \langle\psi| - |\psi\rangle \langle\psi| \hat{H} = \hat{H}\hat{\rho} - \hat{\rho}\hat{H} = [\hat{H}, \hat{\rho}]\end{aligned} \tag{151}$$

从 Master 方程到 Heisenberg 方程

设在 Schrödinger 绘景下，密度算符 $\hat{\rho}$ 承担全部演化，力学量算符 $\hat{A}$ 与时间无关

$$\begin{aligned}i\hbar \frac{\partial \langle\hat{A}\rangle}{\partial t} =& i\hbar \frac{\partial\, \mathrm{tr}\left(\hat{\rho}\hat{A}\right)}{\partial t} = \mathrm{tr}\left[i\hbar \left(\frac{\partial \hat{\rho}}{\partial t}\right) \hat{A}\right] = \mathrm{tr}\left([\hat{H}, \hat{\rho}]\hat{A}\right) = \mathrm{tr}\left(\hat{H}\hat{\rho}\hat{A} - \hat{\rho}\hat{H}\hat{A}\right) \\ =& \mathrm{tr}\left(\hat{\rho}\hat{A}\hat{H} - \hat{\rho}\hat{H}\hat{A}\right) = \mathrm{tr}\left(\hat{\rho}[\hat{A}, \hat{H}]\right) = \langle[\hat{A}, \hat{H}]\rangle\end{aligned} \tag{152}$$

转到 Heisenberg 绘景

$$i\hbar \frac{\partial \hat{A}}{\partial t} = [\hat{A}, \hat{H}] \tag{153}$$

3.2 电磁场密度算符的 Fock 态表示

将电磁场密度算符按 Fock 态展开

$$\rho = \sum_{n,m=0}^{\infty} |n\rangle \langle n| \rho |m\rangle \langle m| = \sum_{n,m=0}^{\infty} \rho_{nm} |n\rangle \langle m| = \sum_{n=0}^{\infty} \rho_{nn} |n\rangle \langle n| + \sum_{n\neq m}^{\infty} \rho_{nm} |n\rangle \langle m| \tag{154}$$

对角元 $P_n = \rho_{nn} \geq 0$ 表示光场的光子数分布，非对角元 $\rho_{nm}(n \neq m)$ 一般是复数，描述光场的位相信息。若只关注光子数的分布情况，可忽略其中非对角项，得到密度算符的约化形式

$$\rho = \sum_{n=0}^{\infty} \rho_{nn} |n\rangle \langle n| = \sum_{n=0}^{\infty} P_n |n\rangle \langle n| \tag{155}$$

Examples

1. **相干光** $|\alpha\rangle$

$$\rho = |\alpha\rangle \langle\alpha| = e^{-|\alpha|^2} \sum_{n,m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{(\alpha^*)^m}{\sqrt{m!}} |n\rangle \langle m| \tag{156}$$

平均光子数

$$\langle n\rangle = \langle\alpha| a^\dagger a |\alpha\rangle = |\alpha|^2 \tag{157}$$

分布函数

$$P_n = \rho_{nn} = \langle n| \rho |n\rangle = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = e^{-\langle n\rangle} \frac{\langle n\rangle^n}{n!} \tag{158}$$

2. **压缩光** $|\xi\rangle = S(\xi)|0\rangle$，$S(\xi) = \exp\left(\frac{1}{2}\xi^* a^2 - \frac{1}{2}\xi a^{\dagger 2}\right)$，$\xi = re^{i\theta}$

$$\rho = |\xi\rangle \langle\xi| \tag{159}$$

平均光子数

$$\langle n\rangle = \mathrm{tr}\left(\rho a^\dagger a\right) = \sinh^2 r \tag{160}$$

分布函数

$$P_n = \frac{(2n)!}{2^{2n}(n!)^2} \frac{\tanh^{2n} r}{\cosh r} \tag{161}$$

3. **热光**

$$\rho = \frac{\exp(-\mathcal{H}/k_B T)}{\mathrm{tr}[\exp(-\mathcal{H}/k_B T)]} = \sum_n \left[1 - \exp\left(-\frac{\hbar\omega}{k_B T}\right)\right] \exp\left(-\frac{n\hbar\omega}{k_B T}\right) |n\rangle \langle n| \tag{162}$$

平均光子数

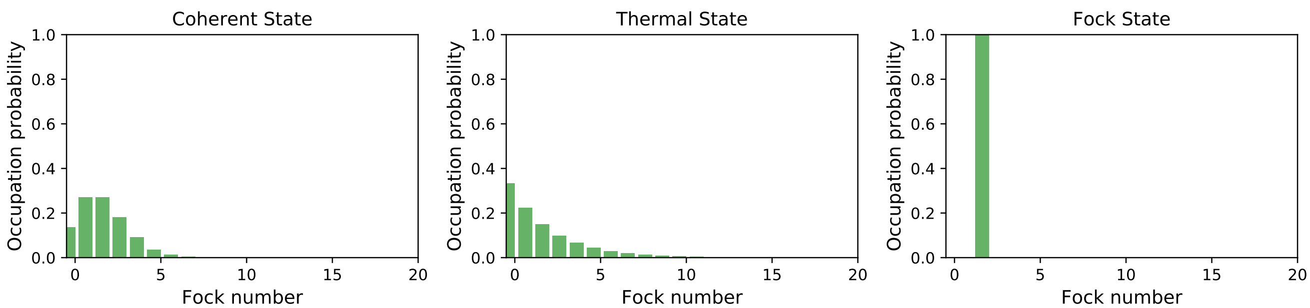
$$\langle n\rangle = \mathrm{tr}\left(a^\dagger a\rho\right) = \left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right]^{-1} \tag{163}$$

$$\rho = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} |n\rangle \langle n| \tag{164}$$

分布函数

$$P_n = \rho_{nn} = \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}} \tag{165}$$

```
N = 20
rho_coherent = coherent_dm(N, np.sqrt(2))
rho_thermal = thermal_dm(N, 2)
rho_fock = fock_dm(N, 2)
fig, axes = plt.subplots(1, 3, figsize=(12,3))
plot_fock_distribution(rho_coherent, fig=fig, ax=axes[0], title='Coherent State')
plot_fock_distribution(rho_thermal, fig=fig, ax=axes[1], title='Thermal State')
plot_fock_distribution(rho_fock, fig=fig, ax=axes[2], title='Fock State')
fig.tight_layout()
plt.show()
```



3.3 电磁场密度算符的相干态表示

由于相干态互不正交，使得一段时间内未被用来作为表象的基底考虑。然而能否作为基底的必要条件不是正交性，而是完备性。相干态具有超完备性，自然可以作为表象的基底。

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha=1 \tag{166}$$

使用相干态表象的优点:

1. 可将量子理论中"力学量期望值的计算"变为"普通复变函数的积分计算";
2. 可将"量子算符方程"过渡到"复变函数的运动方程"。

3.3.1 R 函数

R 表示是密度算符在相干态中的一般表示。由完备性条件 Eq.(166)，密度算符按相干态展开的一般形式为

$$\rho=\iint\frac{\mathrm{d}^2\alpha}{\pi}\frac{\mathrm{d}^2\beta}{\pi}|\alpha\rangle\langle\alpha|\rho|\beta\rangle\langle\beta|=\iint\frac{\mathrm{d}^2\alpha}{\pi}\frac{\mathrm{d}^2\beta}{\pi}|\alpha\rangle\langle\beta|\langle\alpha|\rho|\beta\rangle \tag{167}$$

定义 **R 函数**

$$R(\alpha^*,\beta)=\langle\alpha|\rho|\beta\rangle\,e^{\frac{1}{2}(|\alpha|^2+|\beta|^2)} \tag{168}$$

密度算符可表示为

$$\rho=\iint\frac{\mathrm{d}^2\alpha}{\pi}\frac{\mathrm{d}^2\beta}{\pi}|\alpha\rangle\langle\beta|\,R(\alpha^*,\beta)e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} \tag{169}$$

3.3.2 P 函数

密度算符按相干态展开的一般形式

$$\rho = \iint \frac{\mathrm{d}^2\alpha}{\pi}\frac{\mathrm{d}^2\beta}{\pi}\left|\alpha\right\rangle\left\langle\beta\right|\left\langle\alpha\right|\rho\left|\beta\right\rangle \tag{170}$$

若我们只关注光子数的分布情况，可忽略非对角元，得到密度算符的约化形式

$$\rho = \int P(\alpha,\alpha^*)\left|\alpha\right\rangle\left\langle\alpha\right|\mathrm{d}^2\alpha \tag{171}$$

其中 $P(\alpha,\alpha^*)$ 称为 **P 函数**。P 函数具有以下性质

1. 由密度算符的厄米性，可知其对角元均为实数，故 $P(\alpha,\alpha^*)$ 为实数；
2. 归一化

$$\mathrm{tr}(\rho) = \int P(\alpha,\alpha^*)\mathrm{d}^2\alpha = 1 \tag{172}$$

由此可见，实数化、归一化的 P 函数类似于一个某态在相干态基底上的概率分布函数。然而与后者不同的是，对于有些量子态，P 函数可能是负的或非常奇异的。因此，我们将 P 函数称为**准概率分布函数** (Quasi-probability distribution)，将 P 函数为负或者非常奇异的量子态称为**非经典态**。

以下讨论引入 P 函数的物理意义。考虑正序算符 (Normal ordered operator) $O_N(a,a^\dagger)$，

$$O_N(a,a^\dagger) = \sum_n\sum_m c_{nm}(a^\dagger)^n a^m \tag{173}$$

计算期望值

$$\begin{aligned}\langle O_N(a,a^\dagger)\rangle &= \mathrm{tr}\big[\rho O_N(a,a^\dagger)\big] = \sum_n\sum_m c_{nm}\,\mathrm{tr}\big[\rho(a^\dagger)^n a^m\big] = \sum_n\sum_m c_{nm}\,\mathrm{tr}\big[a^m\rho(a^\dagger)^n\big] \\ &= \sum_n\sum_m c_{nm}\,\mathrm{tr}\left[\int P(\alpha,\alpha^*)a^m\left|\alpha\right\rangle\left\langle\alpha\right|(a^\dagger)^n\mathrm{d}^2\alpha\right] \\ &= \int\left[\sum_n\sum_m c_{nm}(\alpha^*)^n\alpha^m\right]P(\alpha,\alpha^*)\mathrm{d}^2\alpha \\ &= \int O_N(\alpha,\alpha^*)P(\alpha,\alpha^*)\mathrm{d}^2\alpha\end{aligned} \tag{174}$$

可见，在相干态中用 P 函数对正序算符均值进行计算时，可以化为普通函数的积分运算。因为在量子光学中我们讨论的是场的变换，而场算符与 a 和 $a^\dagger$ 联系紧密，因此使用 P 函数计算正序算符的均值是十分方便的。

P 函数的定义由约化密度算符给出，接下来讨论如何从密度算符得到 P 函数。

1. **从算符 δ 函数出发**

   给出正序算符 δ 函数的定义

$$\delta(\alpha^*-a^\dagger)\delta(\alpha-a) = \frac{1}{\pi^2}\int\exp\big[-\beta(\alpha^*-a^\dagger)\big]\exp[\beta^*(\alpha-a)]\mathrm{d}^2\beta \tag{175}$$

   首先证明两个性质：

   (a) 性质一：对于相干态 $\left|\gamma\right\rangle$，有 $\left\langle\gamma\right|\delta\left(\alpha^*-a^\dagger\right)\delta(\alpha-a)\left|\gamma\right\rangle = \delta^2(\alpha-\gamma)$，证明如下

$$\begin{aligned}\left\langle\gamma\right|\delta\left(\alpha^*-a^\dagger\right)\delta(\alpha-a)\left|\gamma\right\rangle &= \left\langle\gamma\right|\frac{1}{\pi^2}\int\exp\big[-\beta(\alpha^*-a^\dagger)\big]\exp[\beta^*(\alpha-a)]\mathrm{d}^2\beta\left|\gamma\right\rangle \\ &= \frac{1}{\pi^2}\int\exp[-\beta(\alpha^*-\gamma^*)]\exp[\beta^*(\alpha-\gamma)]\mathrm{d}^2\beta\end{aligned} \tag{176}$$

令 $\alpha-\gamma=(k_y-ik_x)/2$，$\beta=x+iy$，则

$$\begin{aligned}\langle\gamma|\,\delta\left(\alpha^*-a^\dagger\right)\delta(\alpha-a)\,|\gamma\rangle=&\frac{1}{\pi^2}\int\exp[-i(k_xx+k_yy)]\mathrm{d}x\mathrm{d}y=\delta(k_x)\delta(k_y)\\=&\delta[\mathrm{Im}(\alpha-\gamma)]\delta[\mathrm{Re}(\alpha-\gamma)]=\delta^2(\alpha-\gamma)\end{aligned}\tag{177}$$

(b) 性质二：$(a^\dagger)^na^m=\int(\alpha^*)^n\alpha^m\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\mathrm{d}^2\alpha$，证明如下

$$\begin{aligned}\langle\gamma|\int(\alpha^*)^n\alpha^m\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\mathrm{d}^2\alpha\,|\gamma\rangle=&\int(\alpha^*)^n\alpha^m\delta(\alpha^*-\gamma^*)\delta(\alpha-\gamma)\mathrm{d}^2\alpha\\=&(\gamma^*)^n\gamma^m=\langle\gamma|\,(a^\dagger)^na^m\,|\gamma\rangle=\langle(a^\dagger)^na^m\rangle\end{aligned}\tag{178}$$

故

$$(a^\dagger)^na^m=\int(\alpha^*)^n\alpha^m\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\mathrm{d}^2\alpha\tag{179}$$

由此，考虑正序算符的期望

$$\begin{aligned}\langle O_N(a,a^\dagger)\rangle=&\mathrm{tr}\left[\sum_{n,m}c_{nm}(a^\dagger)^na^m\rho\right]\\=&\mathrm{tr}\left(\sum_{n,m}c_{nm}\int\mathrm{d}^2\alpha\left[(\alpha^*)^n\alpha^m\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\rho\right]\right)\\=&\int\mathrm{d}^2\alpha\sum_{n,m}c_{nm}(\alpha^*)^n\alpha^m\,\mathrm{tr}\big[\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\rho\big]\\=&\int\mathrm{d}^2\alpha O_N(\alpha,\alpha^*)\,\mathrm{tr}\big[\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\rho\big]\end{aligned}\tag{180}$$

故 P 函数表达式

$$P(\alpha,\alpha^*)=\mathrm{tr}\big[\rho\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\big]\tag{181}$$

2. **从 Fourier 变换出发**

设 $|-\beta\rangle$ 和 $|\beta\rangle$ 均为本征值为 $-\beta$ 和 β 的相干态。利用相干态的非正交性质

$$\langle\beta|\alpha\rangle=e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)+\beta^*\alpha}\tag{182}$$

计算矩阵元

$$\begin{aligned}\langle-\beta|\,\rho\,|\beta\rangle=&\int P(\alpha,\alpha^*)\,\langle-\beta|\alpha\rangle\,\langle\alpha|\beta\rangle\,\mathrm{d}^2\alpha\\=&e^{-|\beta|^2}\int P(\alpha,\alpha^*)e^{-|\alpha|^2}e^{\beta\alpha^*-\beta^*\alpha}\mathrm{d}^2\alpha\end{aligned}\tag{183}$$

则

$$\langle-\beta|\,\rho\,|\beta\rangle\,e^{|\beta|^2}=\int\left[P(\alpha,\alpha^*)e^{-|\alpha|^2}\right]e^{\beta\alpha^*-\beta^*\alpha}\mathrm{d}^2\alpha\tag{184}$$

观察发现，上式是 $\langle-\beta|\,\rho\,|\beta\rangle\,e^{|\beta|^2}$ 与 $P(\alpha,\alpha^*)e^{-|\alpha|^2}$ 间的 Fourier 变换关系。由 Fourier 正逆变换的角度很容易得到 P 函数关于密度算符的另一表达式

$$P(\alpha,\alpha^*)=\frac{e^{|\alpha|^2}}{\pi^2}\int\langle-\beta|\,\rho\,|\beta\rangle\,e^{|\beta|^2}e^{-\beta\alpha^*+\beta^*\alpha}\mathrm{d}^2\beta\tag{185}$$

Examples

1. 热场的 P 函数

$$\rho=\sum_n P_n\left|n\right\rangle\left\langle n\right|=\sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\left|n\right\rangle\left\langle n\right| \tag{186}$$

$$\left\langle -\beta\right|\rho\left|\beta\right\rangle=\sum_n\frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}\left\langle -\beta|n\right\rangle\left\langle n|\beta\right\rangle=\frac{e^{-|\beta|^2}}{1+\langle n\rangle}\exp\left(\frac{-|\beta|^2}{1+1/\langle n\rangle}\right) \tag{187}$$

故 P 函数

$$P(\alpha,\alpha^*)=\frac{e^{|\alpha|^2}}{\pi^2(1+\langle n\rangle)}\int\exp\left(\frac{-|\beta|^2}{1+1/\langle n\rangle}\right)\exp(-\beta\alpha^*+\beta^*\alpha)\mathrm{d}^2\beta=\frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle} \tag{188}$$

2. 相干态 $|\alpha_0\rangle$ 的 P 函数

$$P(\alpha,\alpha^*)=\mathrm{tr}\Big[\left|\alpha_0\right\rangle\left\langle\alpha_0\right|\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\Big]=\left\langle\alpha_0\right|\delta(\alpha^*-a^\dagger)\delta(\alpha-a)\left|\alpha_0\right\rangle=\delta^2(\alpha-\alpha_0) \tag{189}$$

3. Fock 态 $|n\rangle$ 的 P 函数

$$\left\langle -\beta\right|\rho\left|\beta\right\rangle=\left\langle -\beta|n\right\rangle\left\langle n|\beta\right\rangle=e^{-|\beta|^2}\frac{(-1)^n|\beta|^{2n}}{n!} \tag{190}$$

$$\begin{aligned}P(\alpha,\alpha^*)=&\frac{(-1)^n e^{|\alpha|^2}}{\pi^2 n!}\int|\beta|^{2n}e^{-\beta\alpha^*+\beta^*\alpha}\mathrm{d}^2\beta\\=&\frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial(\alpha^*)^n}\frac{1}{\pi^2}\int e^{-\beta\alpha^*+\beta^*\alpha}\mathrm{d}^2\beta\\=&\frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial(\alpha^*)^n}\delta^{(2)}(\alpha)\end{aligned} \tag{191}$$

P 函数数是二维 δ 函数的高阶导数，这比二维 δ 函数具有更强的奇异性。此外，由于 δ 函数具有上升沿与下降沿，因此其高阶导数不会恒为正值，这正是我们将 P 函数称为准概率分布函数的原因。

3.3.3 Q 函数

由前面的讨论可知，P 函数只在对正序算符的均值计算中有优势。对于反序算符 (Antinormal ordered operator) $O_A(a,a^\dagger)=\sum_{n,m}c_{nm}a^n(a^\dagger)^m$ ，与 P 函数类似，我们可以尝试定义 Q 函数

$$Q(\alpha,\alpha^*)=\mathrm{tr}\left[\rho\delta(\alpha-a)\delta(\alpha^*-a^\dagger)\right]=\frac{1}{\pi}\left\langle\alpha\right|\rho\left|\alpha\right\rangle \tag{192}$$

利用相干态的超完备性

$$\frac{1}{\pi}\int\left|\alpha\right\rangle\left\langle\alpha\right|\mathrm{d}^2\alpha=1 \tag{193}$$

有

$$\begin{aligned}Q(\alpha,\alpha^*)=&\frac{1}{\pi}\mathrm{tr}\left\{\int\mathrm{d}^2\alpha'\left[\rho\delta(\alpha-a)\left|\alpha'\right\rangle\left\langle\alpha'\right|\delta(\alpha^*-a^\dagger)\right]\right\}\\=&\frac{1}{\pi}\mathrm{tr}\left\{\int\mathrm{d}^2\alpha'\left[\rho\delta(\alpha-\alpha')\left|\alpha'\right\rangle\left\langle\alpha'\right|\delta(\alpha^*-\alpha'^*)\right]\right\}\\=&\frac{1}{\pi}\mathrm{tr}(\rho\left|\alpha\right\rangle\left\langle\alpha\right|)=\frac{1}{\pi}\left\langle\alpha\right|\rho\left|\alpha\right\rangle\end{aligned} \tag{194}$$

其正比于密度算符在相干态表象中的矩阵元。

Q 函数和 P 函数之间可进行转换

$$Q(\alpha,\alpha^*)=\frac{1}{\pi}\int P(\alpha',\alpha'^*)e^{-|\alpha'-\alpha|^2}\mathrm{d}^2\alpha \tag{195}$$

Examples

1. 相干态 $|\beta\rangle$ 的 Q 函数

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}|\langle\alpha|\beta\rangle|^2 = \frac{1}{\pi}e^{-|\alpha-\beta|^2} \tag{196}$$

相干态 $|\beta\rangle$ 的 Q 函数在复平面上，是以 β 为中心，实部和虚部的方差均为 $\frac{1}{2}$ 的高斯函数。

2. Fock 态 $|n\rangle$ 的 Q 函数

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}|\langle\alpha|n\rangle|^2 = \frac{|\alpha|^{2n}}{\pi n!}e^{-|\alpha|^2} \tag{197}$$

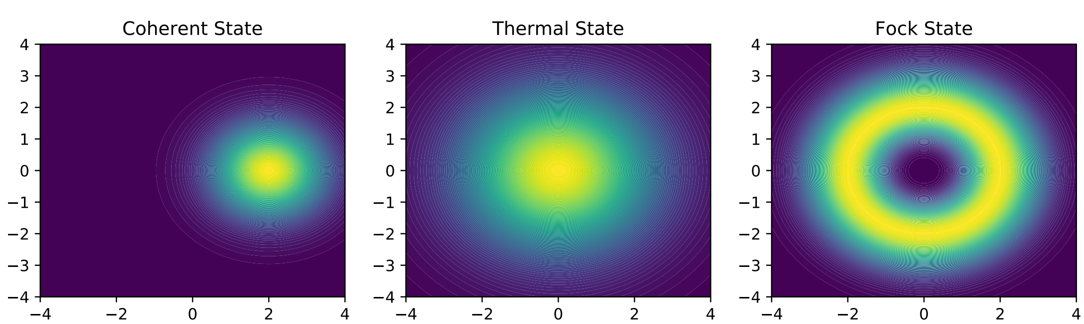
3. 热光场

$$\begin{aligned} Q(\alpha) &= \sum_n \frac{1}{\pi}P_n \langle\alpha|n\rangle\langle n|\alpha\rangle \\ &= \sum_n \frac{1}{\pi}\frac{(\bar{n})^n}{(\bar{n}+1)^{n+1}}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \\ &= \frac{1}{\pi(\bar{n}+1)}e^{-|\alpha|^2}e^{\frac{\bar{n}|\alpha|^2}{\bar{n}+1}} \\ &= \frac{1}{\pi(\bar{n}+1)}e^{-\frac{|\alpha|^2}{\bar{n}+1}} \end{aligned} \tag{198}$$

热光场的 Q 函数是一个以坐标原点为中心，实部和虚部的方差均为 $(\bar{n}+1)/2$ 的高斯函数。

```python
N = 20
xvec = np.linspace(-4,4,100)
rho_coherent = coherent_dm(N, np.sqrt(2))
rho_thermal = thermal_dm(N, 2)
rho_fock = fock_dm(N, 2)
Q_coherent = qfunc(rho_coherent, xvec, xvec)
Q_thermal = qfunc(rho_thermal, xvec, xvec)
Q_fock = qfunc(rho_fock, xvec, xvec)

fig, axes = plt.subplots(1, 3, figsize=(12,3))
cont0 = axes[0].contourf(xvec, xvec, Q_coherent, 100)
lbl0 = axes[0].set_title("Coherent State")
cont1 = axes[1].contourf(xvec, xvec, Q_thermal, 100)
lbl1 = axes[1].set_title("Thermal State")
cont2 = axes[2].contourf(xvec, xvec, Q_fock, 100)
lbl2 = axes[2].set_title("Fock State")
plt.show()
```



3.3.4 Wigner 函数

对称序 (也称 Weyl 排序) 算符 (Symmetrized ordered operator) $O_S = \{(a^\dagger)^n, a^m\}$ 【待施工】

$$W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int e^{\beta^*\alpha-\beta\alpha^*}\,\mathrm{tr}\left(\rho e^{\beta a^\dagger-\beta^* a}\right)\mathrm{d}^2\beta = \frac{2}{\pi}\,\mathrm{tr}\left[(-1)^{a^\dagger a}D^\dagger(\alpha)\rho D(\alpha)\right] \tag{199}$$

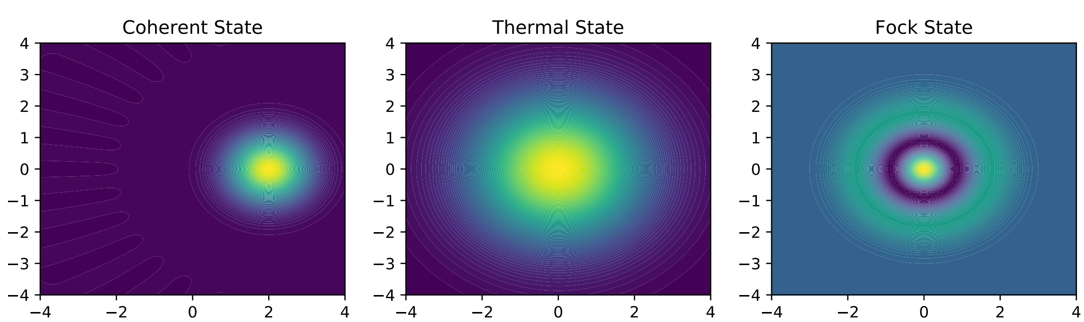
相干态 $|\beta\rangle$ 的 Wigner 函数

$$W(\alpha,\alpha^*) = \frac{2}{\pi}e^{-2|\alpha-\beta|^2} \tag{200}$$

是以 β 为中心，实部和虚部的方差均为 $\frac{1}{4}$ 的高斯函数。

```
N = 20
xvec = np.linspace(-4,4,100)
rho_coherent = coherent_dm(N, np.sqrt(2))
rho_thermal = thermal_dm(N, 2)
rho_fock = fock_dm(N, 2)
W_coherent = wigner(rho_coherent, xvec, xvec)
W_thermal = wigner(rho_thermal, xvec, xvec)
W_fock = wigner(rho_fock, xvec, xvec)

fig, axes = plt.subplots(1, 3, figsize=(12,3))
cont0 = axes[0].contourf(xvec, xvec, W_coherent, 100)
lbl0 = axes[0].set_title("Coherent State")
cont1 = axes[1].contourf(xvec, xvec, W_thermal, 100)
lbl1 = axes[1].set_title("Thermal State")
cont2 = axes[2].contourf(xvec, xvec, W_fock, 100)
lbl2 = axes[2].set_title("Fock State")
plt.show()
```



3.3.5 小结：三种分布函数间的变换

- **P → Q**

$$Q(\alpha) = \frac{1}{\pi}\int d^2\alpha' P(\alpha')\, e^{-|\alpha-\alpha'|^2} \tag{201}$$

- **P → W**

$$W(\alpha) = \frac{2}{\pi}\int d^2\alpha' P(\alpha')\, e^{-2|\alpha-\alpha'|^2} \tag{202}$$

- **W → Q**

$$Q(\alpha) = \frac{2}{\pi}\int d^2\alpha' W(\alpha')\, e^{-2|\alpha-\alpha'|^2} \tag{203}$$

4 电磁场的相干性

4.1 光子探测与量子场关联函数

场算符 $\vec{E}(\vec{r},t)$ 可分为正频部分和负频部分

$$\vec{E}(\vec{r},t)=\vec{E}^{(+)}(\vec{r},t)+\vec{E}^{(-)}(\vec{r},t) \tag{204}$$

其中

$$\vec{E}^{(+)}(\vec{r},t)=\sum_{\vec{k}}\hat{\epsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}a_{\vec{k}}e^{-i\omega_k t+i\vec{k}\cdot\vec{r}} \tag{205}$$

$$\vec{E}^{(-)}(\vec{r},t)=\sum_{\vec{k}}\hat{\epsilon}_{\vec{k}}\mathcal{E}_{\vec{k}}a^{\dagger}_{\vec{k}}e^{i\omega_k t-i\vec{k}\cdot\vec{r}} \tag{206}$$

为了方便计算，以下我们假设电磁场是线性偏振的，故引入标量 $E^{(+)}=\hat{\epsilon}\cdot\vec{E}^{(+)}(\vec{r},t)$ 及 $E^{(-)}=\hat{\epsilon}\cdot\vec{E}^{(-)}(\vec{r},t)$。

光子探测过程是探测器原子吸收一个光子并发射一个光电子的过程，光场的量子特性通过光电效应的量子性体现出来。将基态原子放置于电磁场中 $\vec{r}$ 位置。在光子探测过程中，只有光场的正频部分起作用，从而导致在探测过程中在 $E^{(+)}$ 和 $E^{(-)}$ 之间引入了不对称性，实际探测的是场的**正频部分**算符而不是整个场算符。探测器在位置 $\vec{r}$ 在 t 到 $t+\mathrm{d}t$ 时间内吸收一个光子产生跃迁的概率

$$w_1(\vec{r},t)=|\langle f|E^{(+)}(\vec{r},t)|i\rangle|^2 \tag{207}$$

由于我们一般不测量电磁场末态，而只关心光子计数率，故对所有末态求和

$$\begin{aligned}w_1(\vec{r},t)&=\sum_f|\langle f|E^{(+)}(\vec{r},t)|i\rangle|^2\\&=\sum_f\langle i|E^{(-)}(\vec{r},t)|f\rangle\langle f|E^{(+)}(\vec{r},t)|i\rangle\\&=\langle i|E^{(-)}(\vec{r},t)E^{(+)}(\vec{r},t)|i\rangle\end{aligned} \tag{208}$$

通常地，我们也无法准确得知电磁场的初态，故我们求助于统计描述，对所有初态求加权平均

$$w_1(\vec{r},t)=\sum_i P_i\langle i|E^{(-)}(\vec{r},t)E^{(+)}(\vec{r},t)|i\rangle \tag{209}$$

若引入密度算符

$$\rho=\sum_i P_i|i\rangle\langle i| \tag{210}$$

则有

$$w_1(\vec{r},t)=\mathrm{tr}\big[\rho E^{(-)}(\vec{r},t)E^{(+)}(\vec{r},t)\big] \tag{211}$$

定义**一阶关联函数** (First-order correlation funtion)

$$G^{(1)}(x,x')=\mathrm{tr}\left[\rho E^{(-)}(x)E^{(+)}(x')\right]=\langle E^{(-)}(x)E^{(+)}(x')\rangle \tag{212}$$

其中 $x=(\vec{r},t),x'=(\vec{r}',t')$，且 $t'\geq t$。通常我们考虑关联函数具有时间平移不变性的统计平稳电磁场，即

$$G^{(1)}(x,x')=G^{(1)}(\vec{r},\vec{r}';\tau),\qquad \tau=t'-t \tag{213}$$

探测器光子计数率可用 $G^{(1)}$ 表示为

$$w_1(\vec{r},t)=G^{(1)}(\vec{r},\vec{r};\tau=0) \tag{214}$$

接下来考虑在时空点 $x_1=(\vec{r}_1,t_1),x_2=(\vec{r}_2,t_2),(t_2\geq t_1)$ 的两个探测器的联合光子计数率

$$w_2(x_1,x_2)=|\langle f|\,E^{(+)}(x_2)E^{(+)}(x_1)\,|i\rangle\,|^2 \tag{215}$$

对所有末态求和及对所有可能初态求加权平均，得到

$$w_2(x_1,x_2)=\mathrm{tr}\left[\rho E^{(-)}(x_1)E^{(-)}(x_2)E^{(+)}(x_2)E^{(+)}(x_1)\right] \tag{216}$$

定义**二阶关联函数** (Second-order correlation funtion)

$$G^{(2)}(x_1,x_2,x_3,x_4)=\mathrm{tr}\left[\rho E^{(-)}(x_1)E^{(-)}(x_2)E^{(+)}(x_3)E^{(+)}(x_4)\right] \tag{217}$$

更普遍地，n **阶关联函数** (nth-order correlation funtion)

$$G^{(n)}(x_1,\cdots,x_n,x_{n+1},\cdots,x_{2n})=\mathrm{tr}\left[\rho E^{(-)}(x_1)\cdots E^{(-)}(x_n)E^{(+)}(x_{n+1})\cdots E^{(+)}(x_{2n})\right] \tag{218}$$

QuTiP function	Correlation function
`qutip.correlation.correlation` or `qutip.correlation.correlation_2op_2t`	$\langle A(t+\tau)B(t)\rangle$ or $\langle A(t)B(t+\tau)\rangle$
`qutip.correlation.correlation_ss` or `qutip.correlation.correlation_2op_1t`	$\langle A(\tau)B(0)\rangle$ or $\langle A(0)B(\tau)\rangle$
`qutip.correlation.correlation_3op_1t`	$\langle A(0)B(\tau)C(0)\rangle$
`qutip.correlation.correlation_3op_2t`	$\langle A(t)B(t+\tau)C(t)\rangle$
`qutip.correlation.correlation_4op_1t`	$\langle A(0)B(\tau)C(\tau)D(0)\rangle$
`qutip.correlation.correlation_4op_2t`	$\langle A(t)B(t+\tau)C(t+\tau)D(t)\rangle$

定义在位置 $\vec{r}$ 的一阶和二阶**相干度**

$$g^{(1)}(\vec{r},\tau)=\frac{\langle E^{(-)}(\vec{r},t)E^{(+)}(\vec{r},t+\tau)\rangle}{\sqrt{\langle E^{(-)}(\vec{r},t)E^{(+)}(\vec{r},t)\rangle\langle E^{(-)}(\vec{r},t+\tau)E^{(+)}(\vec{r},t+\tau)\rangle}}=\frac{G^{(1)}(\tau)}{G^{(1)}(0)} \tag{219}$$

$$g^{(2)}(\vec{r},\tau)=\frac{\langle E^{(-)}(\vec{r},t)E^{(-)}(\vec{r},t+\tau)E^{(+)}(\vec{r},t+\tau)E^{(+)}(\vec{r},t)\rangle}{\langle E^{(-)}(\vec{r},t)E^{(+)}(\vec{r},t)\rangle\langle E^{(-)}(\vec{r},t+\tau)E^{(+)}(\vec{r},t+\tau)\rangle}=\frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2} \tag{220}$$

考虑单模光场，相干度可化简为

$$g^{(1)}(\tau)=\frac{\langle a^\dagger(t)a(t+\tau)\rangle}{\langle a^\dagger a\rangle} \tag{221}$$

$$g^{(2)}(\tau)=\frac{\langle a^\dagger(t)a^\dagger(t+\tau)a(t+\tau)a(t)\rangle}{\langle a^\dagger a\rangle^2} \tag{222}$$

```
coherence_function_g1()
coherence_function_g2()
```

又 $a(t)=a(0)e^{-i\omega t}$，根据时间平移不变性，得到

$$g^{(1)}(\tau)=\frac{\langle a^\dagger(0)a(\tau)\rangle}{\langle a^\dagger(0)a(0)\rangle}=e^{-i\omega\tau} \tag{223}$$

$$g^{(2)}(\tau)=\frac{\langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle}{\langle a^\dagger a\rangle^2}=\frac{\langle a^\dagger(0)a^\dagger(0)a(0)a(0)\rangle}{\langle a^\dagger a\rangle^2}=g^{(2)}(0) \tag{224}$$

从物理意义上讲，$g^{(2)}(\tau)$ 代表在 $t+\tau$ 时刻对光强或光子数的测量与在 t 时刻的测量的关联。若 $g^{(2)}(\tau)>1$，则意味着先在 t 时刻测量光子，延迟一小段时间 τ 再测量时，测到光子的几率更大了，这称为**聚束**或正相关。若 $g^{(2)}(\tau)<1$，则意味着先在 t 时刻测量光子，延迟一小段时间 τ 再测量时，测到光子的几率变小，这称为**反聚束**或负相关。

Examples

1. 热场

$$P(\alpha,\alpha^*)=\frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle} \tag{225}$$

$$g^{(2)}(0)=\frac{\int P(\alpha,\alpha^*)|\alpha|^4\mathrm{d}^2\alpha}{\left[\int P(\alpha,\alpha^*)|\alpha|^2\mathrm{d}^2\alpha\right]^2}=2 \tag{226}$$

2. 相干态 $|\alpha_0\rangle$

$$P(\alpha,\alpha^*)=\delta^{(2)}(\alpha-\alpha_0) \tag{227}$$

$$g^{(2)}(0)=1 \tag{228}$$

4.2 一阶相干

4.2.1 Young's 双缝实验

$$0\le|g^{(1)}(\tau)|\le 1 \tag{229}$$

$$E^{(+)}(\vec{r},t)=K_1E^{(+)}(\vec{r}_1,t-t_1)+K_2E^{(+)}(\vec{r}_2,t-t_2) \tag{230}$$

$$\begin{aligned}\langle I(\vec{r},t)\rangle=&\mathrm{tr}\big[\rho E^{(-)}(\vec{r},t)E^{(+)}(\vec{r},t)\big]\\=&|K_1|^2\,\mathrm{tr}\big[\rho E^{(-)}(\vec{r}_1,t-t_1)E^{(+)}(\vec{r}_1,t-t_1)\big]+|K_2|^2\,\mathrm{tr}\big[\rho E^{(-)}(\vec{r}_2,t-t_2)E^{(+)}(\vec{r}_2,t-t_2)\big]\\&+2\,\mathrm{Re}\left\{K_1^*K_2\,\mathrm{tr}\big[\rho E^{(-)}(\vec{r}_1,t-t_1)E^{(+)}(\vec{r}_2,t-t_2)\big]\right\}\\=&|K_1|^2G^{(1)}(\vec{r}_1,\vec{r}_1;t-t_1.t-t_1)+|K_2|^2G^{(1)}(\vec{r}_2,\vec{r}_2;t-t_2,t-t_2)\\&+2\,\mathrm{Re}\left[K_1^*K_2G^{(1)}(\vec{r}_1.\vec{r}_2;t-t_1,t-t_2)\right]\\=&|K_1|^2G^{(1)}(\vec{r}_1,\vec{r}_1;0)+|K_2|^2G^{(1)}(\vec{r}_2,\vec{r}_2;0)+2\,\mathrm{Re}\left[K_1^*K_2G^{(1)}(\vec{r}_1,\vec{r}_2;\tau)\right]\end{aligned} \tag{231}$$

$$\langle I^{(i)}(\vec{r})\rangle=|K_i|^2G^{(1)}(\vec{r}_i,\vec{r}_i;0) \tag{232}$$

$$\langle I(\vec{r},t)\rangle=\langle I^{(1)}(\vec{r})\rangle+\langle I^{(2)}(\vec{r})\rangle+2\sqrt{\langle I^{(1)}(\vec{r})\rangle\langle I^{(2)}(\vec{r})\rangle}\,\mathrm{Re}\left[g^{(1)}(\vec{r}_1,\vec{r}_2;\tau)\right] \tag{233}$$

令

$$g^{(1)}(\vec{r}_1,\vec{r}_2;\tau)=|g^{(1)}(\vec{r}_1,\vec{r}_2;\tau)|\exp[i\alpha(\vec{r}_1,\vec{r}_2;\tau)-i\omega_0\tau] \tag{234}$$

$$\langle I(\vec{r},t)\rangle=\langle I^{(1)}(\vec{r})\rangle+\langle I^{(2)}(\vec{r})\rangle+2\sqrt{\langle I^{(1)}(\vec{r})\rangle\langle I^{(2)}(\vec{r})\rangle}|g^{(1)}(\vec{r}_1,\vec{r}_2;\tau)|\cos\left[\alpha(\vec{r}_1,\vec{r}_2;\tau)-\omega_0\tau\right] \tag{235}$$

$$U=\frac{\langle I(\vec{r})\rangle_{\max}-\langle I(\vec{r})\rangle_{\min}}{\langle I(\vec{r})\rangle_{\max}+\langle I(\vec{r})\rangle_{\min}}=\frac{2\sqrt{\langle I^{(1)}(\vec{r})\rangle\langle I^{(2)}(\vec{r})\rangle}}{\langle I^{(1)}(\vec{r})\rangle+\langle I^{(2)}(\vec{r})\rangle}|g^{(1)}(\vec{r}_1,\vec{r}_2;\tau)| \tag{236}$$

功率谱

$$S(\vec{r},\omega)=\frac{1}{2\pi}\int_0^\infty e^{i\omega t}G^{(1)}(\vec{r},\vec{r},t)\mathrm{d}\tau \tag{237}$$

$$G^{(1)}(\vec{r},\omega)=\int_0^\infty e^{-i\omega t}S(\vec{r},\omega)\mathrm{d}\omega \tag{238}$$

4.2.2 单光子双缝干涉

4.3 二阶相干

4.3.1 Hanbury-Brown-Twiss 实验

4.3.2 光子反群聚效应

4.3.3 零拍混频 (homodyne) 测量

弱信号 a 与强参考光 b 同频，有相同的偏振且都相位锁定

$$c = \sqrt{T}a + i\sqrt{1-T}b \tag{239}$$

$$d = i\sqrt{1-T}a + \sqrt{T}b \tag{240}$$

反射和投射波之间有 $\pi/2$ 的相移

$$\langle c^\dagger c\rangle = T\langle a^\dagger a\rangle + (1-T)\langle b^\dagger b\rangle - i\sqrt{T(1-T)}\langle ab^\dagger - a^\dagger b\rangle \tag{241}$$

$$\langle d^\dagger d\rangle = T\langle b^\dagger b\rangle + (1-T)\langle a^\dagger a\rangle + i\sqrt{T(1-T)}\langle ab^\dagger - a^\dagger b\rangle \tag{242}$$

假设参考光 b 处于相干态光场 $|\beta_l\rangle$，$\beta_l = |\beta_l|e^{i\phi_l}$，则

$$\langle c^\dagger c\rangle = T\langle a^\dagger a\rangle + (1-T)|\beta_l|^2 + 2\sqrt{T(1-T)}|\beta_l|\left\langle X\left(\phi_l + \frac{\pi}{2}\right)\right\rangle \tag{243}$$

- **通常零拍混频 (ordinary homodyne) 方案：若 $T \ll R = 1-T$**

$$X\left(\phi_l + \frac{\pi}{2}\right)_{\phi_l = -\frac{\pi}{2},\frac{3\pi}{2}} = X_1 = \frac{1}{2}(a^\dagger + a) \tag{244}$$

$$X\left(\phi_l + \frac{\pi}{2}\right)_{\phi_l = 0} = X_2 = \frac{1}{2}(a^\dagger - a) \tag{245}$$

- **均衡零拍混频 (balanced homodyne) 方案：若 $T = R = 0.5$**

$$\langle n_{cd}\rangle = \langle c^\dagger c - d^\dagger d\rangle = -i\langle a^\dagger b - ab^\dagger\rangle = 2|\beta_l|\langle X(\phi_l + \frac{\pi}{2})\rangle \tag{246}$$

- **测量 Wigner 函数的非平衡零拍混频 (homodyne) 方案**

$$W(\alpha,\alpha^*) = \frac{2}{\pi}\,\mathrm{tr}\left[(-1)^{a^\dagger a}D^\dagger(\alpha)\rho D(\alpha)\right] = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\langle n|D^\dagger(\alpha)\rho D(\alpha)|n\rangle \tag{247}$$

4.4 光电子计数与光子统计

Quantum efficiency η：吸收一个光子产生一个光电子的概率

$$P_m = \sum_n P_m^{(n)}\rho_{nn} = \sum_n \binom{n}{m}\eta^m(1-\eta)^{n-m}\rho_{nn} \tag{248}$$

观察到 m 个光电子的概率与量子场的 P 函数的联系

$$P_m = \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)\frac{(\eta|\alpha|^2)^m}{m!}e^{-\eta|\alpha|^2} \tag{249}$$

- **相干光** $\rho = |\alpha_0\rangle\langle\alpha_0|$ 光场分布函数

$$\rho_{nn} = \frac{\bar{n}^n}{n!}\exp(-\bar{n}) \tag{250}$$

  光子计数分布函数

$$P_n = \frac{(\eta\bar{n})^n}{n!}\exp(-\eta\bar{n}) \tag{251}$$

- **热光** $\rho = Z^{-1}\exp(-H/k_BT)$ 光场分布函数

$$\rho_{nn} = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \tag{252}$$

  光子计数分布函数

$$P_n = \frac{(\eta\bar{n})^n}{(1+\eta\bar{n})^{n+1}} \tag{253}$$

5 原子与光场的相互作用——半经典理论

- **全经典理论**：光和物质均为经典
- **半经典理论**：光是经典，物质是量子
- **半经典理论**：光是量子，物质是经典
- **全量子理论**：光和物质均为量子

5.1 原子与光场的相互作用 Hamiltonian

原子的 Hamiltonian

$$H_{\mathrm{atom}} = \frac{p^2}{2m} + V(\vec{r}) \tag{254}$$

原子与光场相互作用的 Hamiltonian

$$H_{\mathrm{int}} = -\vec{D}\cdot\vec{E} = -q\vec{r}\cdot\vec{E} \tag{255}$$

总 Hamiltonian

$$H = H_{\mathrm{atom}} + H_{\mathrm{int}} \tag{256}$$

半经典近似：原子是量子化的（考虑能级结构），光场看作经典场

$$H_{\mathrm{atom}} = \sum_n \varepsilon_n \left|n\right\rangle\left\langle n\right| \tag{257}$$

5.1.1 偶极近似和半经典近似

忽略光的磁场对原子的作用
忽略波数相关效应 (均匀电场)

5.1.2 不同绘景和旋波近似

二能级原子 + 光

$$H_{\mathrm{atom}} = \varepsilon_g \left|g\right\rangle\left\langle g\right| + \varepsilon_e \left|e\right\rangle\left\langle e\right| = \varepsilon_g + \hbar\omega_a \left|e\right\rangle\left\langle e\right| \tag{258}$$

$$H_{\mathrm{int}} = -q\vec{r}\cdot\vec{E} = -\hbar\left(\Omega_0 \left|e\right\rangle\left\langle g\right| + \Omega_0^* \left|g\right\rangle\left\langle e\right|\right)\cos(\omega t) \tag{259}$$

在 Schrödinger 绘景下的 Hamiltonian

$$H = \hbar\omega_a \left|e\right\rangle\left\langle e\right| + \hbar\Omega_0(\left|g\right\rangle\left\langle e\right| + \left|e\right\rangle\left\langle g\right|)\cos(\omega t) \tag{260}$$

$$\sigma^\dagger = \left|e\right\rangle\left\langle g\right| \qquad\qquad \sigma = \left|g\right\rangle\left\langle e\right| \tag{261}$$

5.2 二能级原子与单模光场

5.2.1 概率幅方法

$$|\psi(t)\rangle = C_a(t)\,|a\rangle + C_b(t)\,|b\rangle \tag{262}$$

其中 C_a 和 C_b 是原子在态 $|a\rangle$ 和 $|b\rangle$ 上的概率。相应的 Schrödinger 方程为

$$\left|\dot{\psi}(t)\right\rangle = -\frac{i}{\hbar}\mathcal{H}\,|\psi(t)\rangle \tag{263}$$

其中

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{264}$$

利用完备性关系 $|a\rangle\langle a| + |b\rangle\langle b| = 1$，可以将 $\mathcal{H}_0$ 改写成

$$\mathcal{H}_0 = (|a\rangle\langle a| + |b\rangle\langle b|)\,\mathcal{H}_0\,(|a\rangle\langle a| + |b\rangle\langle b|) = \hbar\omega_a\,|a\rangle\langle a| + \hbar\omega_b\,|b\rangle\langle b| \tag{265}$$

5.2.2 相互作用绘景

5.3 二能级原子的密度矩阵描述

5.3.1 密度矩阵的运动方程

5.3.2 二能级原子

$$\begin{aligned}\rho = |\psi\rangle\langle\psi| &= [C_a(t)\,|a\rangle + C_b(t)\,|b\rangle]\,[C_a^*(t)\,\langle a| + C_b^*(t)\,\langle b|] \\ &= |C_a|^2\,|a\rangle\langle a| + C_aC_b^*\,|a\rangle\langle b| + C_a^*C_b\,|b\rangle\langle a| + |C_b|^2\,|b\rangle\langle b|\end{aligned} \tag{266}$$

5.4 Maxwell-Schrödinger 方程